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Visualizing the Stochastic Calculus of Option Pricing with Excel and VBA

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Stochastic calculus, part calculus and part statistics, is an integral part of option pricing that can be intimidating. By developing the statistical nature of stochastic processes and introducing Monte Carlo simulation using Microsoft Excel, this paper develops a visualization of how stochastic processes are evaluated using Ito's lemma and integral calculus. Ultimately, the Black-Scholes (1973) option pricing equation is the natural result. [JEL: G10, G12, G13*]*

The pricing of a basic option is essentially the value of being able to eliminate certain future states of the world for a given underlying security. A call option, for example, allows the long position to invest in the underlying security only when the underlying security price is above the strike price; otherwise there is no investment. A stochastic process is a way to mathematically describe a variable that can be in one of many states of the world at a future point in time. By using a stochastic process to describe stock price movement combined with the effective elimination of certain future outcomes for the stock price, we can price a financial option on the stock.

Although the intuition of option pricing is not difficult to grasp, evaluation of the stochastic process governing the underlying security price is another matter. A number of techniques are available to solve for the option price. Although different in their approach, all of these methods are similar in that they must evaluate the future option payoffs, probability weight the option payoffs to calculate the expected future option value, and then discount the expected future option value to obtain the option price. Monte Carlo simulation provides the most intuitive means of option pricing, while the use of stochastic calculus is viewed as the most obtuse approach. In fact, both methods attack the problem in a very similar manner. Consequently, we use Monte Carlo simulation as a means of understanding the stochastic calculus necessary to generate the Black-Scholes (1973) option pricing model.

In Section I, we develop for the reader the statistical knowledge for understanding stochastic processes. In Section II, we introduce Ito's lemma as a means of transforming one stochastic process into another stochastic process. Finally, in Section III, we combine elements of both techniques to generate the Black-Scholes (1973) option pricing model.

I. Statistical Properties of Stochastic Processes

A stochastic process is nothing more than a specification of the probability distribution of the potential future values of a particular variable, although the notation makes the stochastic process appear more complicated. Exhibit 1 lays out a few basic statistical identities concerning the multiplication by a constant and the addition of a constant to a variable that has a probability distribution.

Notice that the addition of a constant does not affect the variance or the standard deviation, but multiplication

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by a constant affects all three statistical measures.

The basic component of many stochastic processes is a normal distribution that has a mean of zero and a variance equal to a small increment of time, symbolized as Δt in discrete time, or dt in continuous time.[1] The normal distribution described above characterizes a Wiener process, and is represented as ΔZ in discrete time (or dZ in continuous time). Notice that a larger time increment, Δt, is associated with a greater variance in the distribution of ΔZ.

A more compact notation for a normal distribution is: *N(mean, variance)*. Consequently, the Wiener process, ΔZ, is distributed $N(0, \Delta t)$. When we multiply ΔZ by a constant, b, the product $b\Delta Z$ becomes distributed $N(b \times 0, b^2\Delta t)$ or $N(0, b^2\Delta t)$ (see Exhibit 1). When we add a constant, $a\Delta t$, to $b\Delta Z$, $a\Delta t + b\Delta Z$ becomes distributed $N(a\Delta t + 0, b^2\Delta t)$ or $N(a\Delta t, b^2\Delta t)$.

$a\Delta t + b\Delta Z$ is an example of a particular stochastic process referred to as arithmetic Brownian motion (ABM). The $a\Delta t$ or sometimes only the a portion is generally called the *mean* or *drift* process of the ABM. The b portion is generally called the *volatility* process of the ABM. Assuming X follows ABM, the current value of X will change by an increment, ΔX, over the next moment in time of duration Δt. ΔX equals $a\Delta t + b\Delta Z$, which is distributed $N(a\Delta t, b^2\Delta t)$.

Geometric Brownian motion (GBM) differs from ABM, because the mean process and the volatility process are affected by the most recent realization of the variable. That is, the mean process and the volatility process for GBM can change with every future increment of time, while under ABM they remain constant through each increment of time (assuming the time increment remains constant).

To generate a GBM process, we define $S(t + \Delta t)$ as the future realization of the variable S at time $t + \Delta t$. Assume t is the current moment in time, making $S(t)$ the most recent realization of the variable S. Next, build the stochastic process for ΔS (i.e., $S(t + \Delta t) - S(t)$), multiply ΔZ by $bS(t)$ creating $bS(t)\Delta Z$, which is distributed $N(bS(t) \times 0, [bS(t)]^2\Delta t)$ or $N(0, [bS(t)]^2\Delta t)$. Add $aS(t)\Delta t$ to $bS(t)\Delta Z$ creating $aS(t)\Delta t + bS(t)\Delta Z$, which is distributed $N(aS(t)\Delta t + 0, [bS(t)]^2\Delta t)$ or $N(aS(t)\Delta t, [bS(t)]^2\Delta t)$. The final distribution is an example of GBM. It is the $S(t)$ parameter (i.e., the most recent realization of the variable) in the mean and variance of ΔS that allows the value of the mean and variance to change through time as new realizations of $S(t)$ emerge.

In general, GBM is of more concern, because it is a common method of modeling stock prices.[2] The a parameter becomes the expected annual return of the stock, while the b parameter represents the annual return volatility. The time increment Δt is expressed in years and is usually a small portion of a year.

To visualize stock price movement through time assuming GBM, an Excel spreadsheet is developed using the Excel commands: =RAND( ) and =NORMINV( ).[3] When we combine the two functions: =NORMINV(RAND( ), x, y), a random number generator is created that draws from a normal distribution defined as $N(x, y^2)$.

Let the current stock price, $S(t)$, be \$10.00 and define the stochastic process for the change in the stock price, $\Delta S = S(t + \Delta t) - S(t)$, as:

$$10\% \times S(t)\Delta t + 20\% \times S(t)\Delta Z$$

which is distributed $N(10\% \times S(t)\Delta t, [20\% \times S(t)]^2\Delta t)$. Further, define Δt as 0.05 of a year and assume the 10% and 20% parameters are per annum. Exhibit 2 illustrates one potential price path for the stock over the next year.

Exhibit 3 provides the Excel commands to generate the price path, which is one trial of a Monte Carlo simulation.

A single simulation trial is sufficient for illustration of GBM, but only after performing many simulation trials does the probability distribution of the future stock price emerge. Further, by performing many simulations using two GBM processes that differ only slightly in the definition of the mean process, an important statistical property appears.

Say, for example, that GBM1 has a mean of $a \times S(t)\Delta t$ and GBM2 has a mean of $c \times S(t)\Delta t$. By taking the average of each set of simulation outcomes generated with the different mean processes and then discounting the average by the appropriate discount rate a or c (as defined within the associated GBM process), the current stock price emerges. Although it may seem a rather miniscule observation, the fact that the current stock price emerges under either set of simulations reflects the underlying fact that the mean process can contain any rate of return when the only concern is to obtain the current stock price.[4]

The distribution of the future payoffs of an option is easy to generate. One simply determines the option's

[1]Continuous time means that the time increment is infinitesimally small. Discrete time means that the time increment is small, but generally measurable (e.g., a second, a minute, a day). For clarity, in this section, we use discrete time increments.

[2]Because of the GBM assumption for stock prices, stock returns will follow ABM.

[3]=RAND( ) generates a random variable with a value between 0 and 1. Given a number between 0 and 1, call it α,=NORMINV(α, mean, standard deviation) returns the numerical outcome of a variable with a normal distribution (mean and standard deviation defined within the function) that has the associated cumulative probability distribution value equal to α.

[4]This is an example of the Martingale property of GBM (see Neftci, 2000).

Exhibit 1. Statistical Relationships When a Constant α is Added to or Multiplied by a Variable with a Probability Distribution

Statistic	$\alpha \times$ (Statistic)	α + Statistic
Mean: μ	$\alpha \times \mu$	$\alpha + \mu$
Variance: σ^2	$\alpha^2 \times (\sigma^2)$	σ^2
Standard Deviation: σ	$\alpha \times \sigma$	σ

Exhibit 2. An Example of Price Movement for One Year

$S(t)$ is set at \$10.00 initially, and $\Delta S = S(t + \Delta t) - S(t)$. Assuming $S(t)$ is known, the stochastic process for ΔS is defined as 10% $S(t)\ \Delta t$ + 20% $S(t)\ \Delta Z$ where Δt is 0.05 years and ΔZ is $N(0, \Delta t)$.

Time in Years	Change in Price	Current Price	Time in Years	Change in Price	Current Price
0.00	N/A	\$10.00	0.55	\$0.0211	\$11.43
0.05	\$0.4355	\$10.44	0.60	\$0.5004	\$11.93
0.10	\$0.5857	\$11.02	0.65	-\$0.1376	\$11.79
0.15	-\$0.3629	\$10.66	0.70	\$0.1387	\$11.93
0.20	\$0.1269	\$10.79	0.75	-\$0.0249	\$11.90
0.25	\$0.2349	\$11.02	0.80	-\$1.0119	\$10.89
0.30	-\$0.2987	\$10.72	0.85	\$0.9964	\$11.89
0.35	\$0.4034	\$11.12	0.90	-\$0.1623	\$11.72
0.40	\$0.3680	\$11.49	0.95	\$0.1626	\$11.89
0.45	-\$0.3894	\$11.10	1.00	\$0.0656	\$11.95
0.50	\$0.3016	\$11.41			

Exhibit 3. Excel Commands for a Single Trial of a Monte Carlo Simulation

To accommodate the length of the given simulation trial, copy cells A7, B7, and C7 to fill in the appropriate number of rows in the spreadsheet. Hitting the F9 key generates a new simulation trial.

	A	B	C
1	Mean:	10%	Annual
2	Volatility:	20%	Annual
3	Time Increment (dt):	0.05	Years
4			
5	Time in Years:	Price Change:	Current Price
6	0.00		\$10.00
7	=A6+\$B\$3	=NORMINV(RAND( ), \$B\$1*C6*\$B\$3, \$B\$2*C6*SQRT(\$B\$3))	=C6+B7
8	Copy Cell A7	Copy Cell B7	Copy Cell C7

value, given the final simulated stock price for each trial. However, taking the average of the future option payoffs and discounting the average to produce the current option price is more difficult. The problem is that the discount rate appropriate for the option payoff is generally different from the discount rate for the underlying stock price process, due to the different (but related) risk characteristics of the option. However, there is one circumstance in which the option and the stock have the same discount rate.

If the risk premiums for the option and the stock are eliminated (that is, we have a risk-neutral environment), then both securities are priced using the risk-free rate. By making the risk-free rate the expected return for the stock within the appropriate GBM process, we create a risk-neutral environment. Although the observation of being able to use any rate of return within the stock price process appears innocuous in regard to finding the current stock price, the ability to create a risk neutral environment proves very helpful in considering the pricing of options.

To implement risk-neutral pricing into our example, we adjust the mean of ΔS to incorporate a risk free rate of 7% per annum: $\Delta S = 7\% \times S(t)\Delta t + 20\% \times S(t)\Delta Z$. Assume there is a need to price a one-year European call option with a strike price of \$13.00. Consequently, any time the stock price simulation finishes above \$13.00, the call option has a final payoff equal to the final stock price minus \$13.00. Otherwise, the value of the call is zero.

To produce multiple simulations manually, we can repeat the single Monte Carlo trial in Exhibit 3 (adjusted to use 7% instead of 10%) numerous times by pressing the F9 key again and again. After each repetition, the final value of S is recorded, and the value of the option payoff is determined. After a sufficient number of repetitions, the distributions of the final stock price and the option payoffs emerge.

For example, if the probability of a given stock price is 25%, then 25% of the simulation trials will produce that particular stock price. Thus, the probability distribution function of future stock prices becomes represented by the frequency of final stock prices within the simulation data. Again, we can discount the average of the final stock prices and the average of the option payoffs by the risk free rate, and find the current values of both the stock and option. However, repeatedly hitting the F9 key and recording the simulated result is not an efficient method for conducting a Monte Carlo analysis. To expedite the Monte Carlo simulation process, we can use Visual Basic (or VBA) code.[5]

To implement the VBA code (assuming the Exhibit 3 simulation trial is already programmed in "Sheet 1" of the Excel spreadsheet), open the Visual Basic editor in Excel either through the [Tools]->[Macro] menu or by pressing <Alt>-<F11>. In the new window VBA code can be written and executed. The smaller window to the left is the Project Explorer window, which provides details about the workbook currently open (if the Project Explorer window does not appear, go to the [View] menu and activate it). To include the VBA program within the workbook, create a module by selecting the name of the current workbook (stoch.xls in this example) within the Project Explorer window and then open the module through the [Insert]->[Module] menu selections. A blank module entitled "Module 1" should appear.

We want to write code that executes the simulation in Worksheet 1 multiple times and records the final outcome of each trial of the simulation in Worksheet 2. The program, which is very simple, appears in Exhibit 4.

To execute the subroutine in the VBA Editor window, click the play button on the toolbar, or activate [Run Macro] in the [Run] menu. Alternatively, to start the subroutine from the Excel workbook, follow the menu sequence: [Tools]->[Macro]->[Macros].

Upon successful execution of the subroutine, Worksheet 2 will contain 1000 observations of the final stock price generated from the Monte Carlo simulation (see Column A of Exhibit 5).

The =MAX( ) function in Column B calculates the payoffs for the one year European call option with a strike price of \$13.00. Taking the average[6] of the column of the final stock prices (approximately \$10.70), the current value of the stock is computed as \$10.70 ÷ 1.07 = \$10.00, as expected. The call option value is the present value of the average of its payoffs (\$0.22 ÷ 1.07), and equals approximately \$0.20.

Because the simulation is generated from a normal distribution, taking the average of many simulation trials is the same as integrating all future outcomes weighted by their associated probability (i.e., taking the expectation of future outcomes). Further, the integral can be represented in terms of a cumulative standard normal distribution,[7] which makes computation very easy via a spreadsheet command or a table.

In the case of stock pricing, there is no benefit in using simulation or an integral, because the current stock price is already known. Option pricing, however, requires either an integral solution or simulation. Because the integral solution requires less computation, it is worthwhile to explore it, but it is equally beneficial to understand how the integral solution is equivalent to the option price obtained via Monte Carlo simulation (that is, being able to visualize what is accomplished by taking the integral).

In the next section, Ito's Lemma (or the calculus of stochastic processes) and another transformation are introduced to provide the necessary building blocks for an integral solution for pricing an option (i.e., the Black-Scholes (1973) option pricing formula). The integral and its equivalency to Monte Carlo simulation are discussed in Section III.

II. Ito's Lemma: The Calculus of Stochastic Processes

Although the goal is to get to an integral solution for option pricing, two transformations are necessary for the calculation of the integral. One transformation involves Ito's Lemma, and the other transformation is simply an application of the results in Exhibit 1.

The calculus necessary to employ Ito's Lemma is very algorithmic. According to Ito's Lemma, one

[5]See Jackson and Staunton (2001) for more details about Excel VBA applications.

[6]The Excel command is =AVERAGE(range).

[7]A standard normal distribution has a mean of zero and a variance of one.

Exhibit 4. Monte Carlo Subroutine in VBA with Explanations

The subroutine can be further modified to be more efficient and to perform more tasks within the subroutine. Here, efficiency and complexity are sacrificed for clarity. The choice of using 1000 trials is arbitrary and is in fact considered a relatively small number of trials, given the abilities of modern computers. Hull (2000) provides other suggestions for increasing the accuracy of the simulation.

VBA Code	Explanation
Sub Monte ( )	Declares subroutine.
Worksheets("Sheet2").Range("a1:a1000").Clear	Clears Worksheet 2 so that it can be used to record simulation data.
For j = 1 to 1000	Starts the loop that will execute 1000 simulation trials.
Worksheets("Sheet1").Calculate	Executes the simulation contained in Worksheet 1.
EndPrc = Worksheets("Sheet1").Range("J27").Value	Creates a variable *EndPrc* that is set equal to the final stock price within the simulation trial. The final stock price is contained in cell J27 of Worksheet 1 in this example.
Worksheets("Sheet2").Cells(j, 1).Value = EndPrc	Takes *EndPrc* and records it in the first column of Worksheet 2. The value for *j* determines which row *EndPrc* is recorded.
Next j	Increments the loop variable by one.
End Sub	Ends subroutine and appears immediately when the subroutine is initially declared.

Exhibit 5. Monte Carlo Simulation Results (Worksheet 2)

	A	B
1	$10.93	$0.00 ←= MAX($13.00 – A1, 0)
2	$10.05	$0.00 ←Copy Cell B1 down to cells B2 through B1000
3	$9.86	$0.00
4	$11.05	$0.00
5	$10.82	$0.00
6	$12.07	$0.00
7	$13.79	$0.79
8	$8.52	$0.00

process can be transformed into another process by taking three partial derivatives and assembling the partial derivatives back into the original stochastic process in a particular manner. The partial derivatives are used merely to transform the original stochastic process into a new stochastic process, and nothing more.

The algorithm is simple and best illustrated with an example in continuous time. Consequently, $\Delta S = a \cdot S(t) \cdot \Delta t + b \cdot S(t) \cdot \Delta Z$ becomes $dS = aSdt + bSdZ$ where dZ is distributed $N(0, dt)$.[8] Let R be a function of S, say $R = ln(S)$, which makes R the continuous time return for S.

To transform the stochastic process for S into a stochastic process for R:

1. Take the following partial derivatives (denoted with subscripts) with respect to the stock price S and time t: $R_S = 1/S$, $R_{SS} = -1/(S^2)$, and $R_t = 0$.

2. Insert the partial derivatives into the original stochastic process as follows to create the new stochastic process: $dR = [\boldsymbol{a}SR_S + 0.5(\boldsymbol{b}\mathbf{S})^2R_{SS} + R_t]dt + \boldsymbol{b}\mathbf{S}R_S\,\mathbf{dZ}$, where boldface indicates the

[8] S is the same as $S(t)$ from before, but more complicated in that S can now change at every instant in time.

mean and volatility portions of the original stochastic process. Substituting the values for the partial derivatives, R is an example of ABM: $dR = [a - 0.5b^2]dt + b\, dZ$.

Notice that the new stochastic process still has the same Wiener process as the original stochastic process. The partial derivatives essentially recalibrate the mean and volatility processes of the original stochastic process to match the new stochastic process. A more formal discussion of Ito's Lemma is provided in the first chapter of Shimko (1992).

It is also worth mentioning again that taking the natural logarithm of the above geometric Brownian motion process for S creates an arithmetic Brownian motion process, R. The benefit of the new ABM process is that its mean and variance do not change with time, although the mean and variance are scaled by the length of the time step, dt. In a GBM process, the mean and the volatility change with every new observation of S, making the parameters of its normal distribution more difficult to work with. In fact, S is considered to be *log-normal*. That is, the natural logarithm of S, the ABM process called R, is normally distributed. The point is: when assessing the probability distribution function of S, it is generally helpful to make a transformation to a normal distribution by taking the natural logarithm of S (i.e., applying Ito's Lemma to create the ABM process, R).

The second transformation is to reduce dR to a standard normal distribution. The distribution of dR is $N([a - 0.5b^2]dt,\ b^2dt)$, which can be transformed into a standard normal distribution by first adding the negative of the mean process $-(a - 0.5b^2)dt$ (i.e., subtracting the mean process). The result $dR - (a - 0.5b^2)dt$ is distributed $N(0,\ b^2dt)$. It is then multiplied by the reciprocal of the standard deviation $b\sqrt{dt}$ (i.e., divided by the standard deviation); the resulting $[dR - (a - 0.5*b^2)*dt] \div b\sqrt{dt}$ is distributed $N(0,\ 1)$.

Notice that the transformation of a normal distribution into a standard normal distribution is simply a matter of applying the results from Exhibit 1. As mentioned in the previous section, standard normal distributions are computationally easy, making this transformation very desirable. With these two transformations, the integral solution for the option price can be determined.

For completeness, it is worth performing the second transformation under the conditions of an integral equation. Let y be distributed $N(\mu,\ \sigma^2)$, or, more formally; the probability of a given observation, y is:

$$\frac{1}{\sigma\sqrt{2\pi}} exp\left(-[y-\mu]^2 \Big/ 2\sigma^2\right)$$

where $exp(\,)$ is the exponential function. We want to evaluate the probability that $y \leq x$:

$$\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-[y-\mu]^2 \Big/ 2\sigma^2\right) dy \tag{1}$$

To transform y into a standard normal distribution, define $q=(y-\mu)/\sigma$. Consequently, $y = q\sigma + \mu$ (i.e., solve the previous equation for y), hence, $dy/dq = \sigma$, making $dy = \sigma\, dq$. In order to complete the change of variables for y and dy in Equation (1), we must adjust the upper and lower limits accordingly. For the upper limit, $y = x = q\sigma + \mu$. Solving for q in terms of x sets the upper limit to $q = (x - \mu)/\sigma$. For the lower limit, $y = -\infty = q\sigma + \mu$. Since σ and μ are constants, and σ is positive, the above equation implies $q = -\infty$. Finally, make the substitution for y and dy in terms of q and dq and adjust the limits in Equation (1).

$$\int_{-\infty}^{\frac{(x-\mu)}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-[q\sigma+\mu-\mu]^2 \Big/ 2\sigma^2\right) \sigma dq = \int_{-\infty}^{\frac{(x-\mu)}{\sigma}} \frac{1}{\sqrt{2\pi}} exp\left(-[q]^2 \Big/ 2\right) dq \tag{2}$$

Notice that the probability density function of q in Equation (2) is the same as that of y, if in Equation (1) μ is set to zero and σ^2 is set to one. In other words, q is distributed as standard normal and its integral can be evaluated using a table. The Black-Scholes (1973) option pricing formula derivation (i.e., the integral solution for the option price) is basically a series of this type of transformation. The next section produces the derivation and relates it back to the Monte Carlo simulation of the previous section.

III. Deriving the Black-Scholes Option Pricing Model (1973)

The Black-Scholes option pricing solution begins with the same stock price process as used in the Monte Carlo simulation. Define the risk-neutralized stock price process in continuous time as $dS = R_F S\, dt + \sigma S\, dZ$, which is distributed $N(R_F\, S\, dt,\ [\sigma S]^2dt)$, where R_F is the appropriate risk-free rate. Relying upon the normally distributed stock price process, and defining the final stock price at option maturity, $dt = T$, as S_T, the equivalent integral solution for a call option with a strike price of X is:

$$\int_{L}^{U} [e^{-R_F T}][max(S_T - X, 0)] \frac{1}{\sigma S\sqrt{2\pi T}} exp\left(-[S - R_F * S * T]^2 \Big/ 2\sigma^2 S^2 T\right) dS$$

$$U = \infty$$

$$L = -\infty \tag{3}$$

Equation (3) is the same as the Monte Carlo simulation solution. The second bracketed term defines the future payoffs of the option, which are then weighted by a probability through the terms after the second bracketed term, and integrated to produce the expected future value of the option. This is the same as taking the average of all the option payoff values from the many simulation trials of the Monte Carlo solution. The expected future option payoffs are discounted using the risk-free rate via the first bracketed term. Again, this is the same procedure applied to the average of the future option payoffs in the Monte Carlo method to produce the option price.

The problem with evaluating Equation (3) is that the substitution of T for dt is not exactly correct, because at every instant in time dt, S can change, which also changes the mean and the variance of the distribution. This is seen in the Monte Carlo option pricing analysis when S is updated (which also updates the mean and the variance of the stock price process) 20 times before reaching the end of an individual simulation trial.[9]

Consequently, the integral in Equation (3) is impossible to evaluate in its current form, because the distribution of the final stock price at time T (i.e., S_T) has a mean and a variance that are not stable throughout the time period T. If we take advantage of the log-normal property of S, however, an integral that can be evaluated emerges.

Define S_T as the current stock price S_0, appreciated by a rate R, resulting in $S_T = S_0 e^R$. Next, transform the price process into a return process, $R = ln(S)$. The new integral can be evaluated, because R is an ABM process that is normally distributed, with a stable mean and variance scaled by T (i.e., the substitution of T for dt can be implemented correctly). Apply the example for Ito's Lemma, $R = ln(S)$, which implies $dR = [R_F - 0.5\sigma^2]dt + \sigma\, dZ$, which is distributed $N([R_F - 0.5\sigma^2]dt,\ \sigma^2 dt)$. Redefining the integral in terms of the return process R and again letting dt equal T creates an equation devoid of components that can change through time.

$$\int_L^U \left[e^{-R_F T}\right]\left[max\left(S_0 e^R - X, 0\right)\right] \frac{1}{\sigma\sqrt{2\pi T}} exp\left(-\left[R - \left(R_F T - \frac{1}{2}\sigma^2 T\right)\right]^2 \Big/ 2\sigma^2 T \right) dR$$

$$U = \infty$$

$$L = -\infty \tag{4}$$

Evaluating the integral is a matter of applying a series of transformations and then exploiting the convenience of a standard normal distribution. The final Black-Scholes solution is: $S_0 N(d_1) - X\, exp(-R_F T)\, N(d_2)$, where d_1 and d_2 are defined in the Appendix (Equations A.8 and A.10), and $N(*)$ represents the cumulative standard normal distribution.

Setting the maturity to one year, the risk free rate to 7% per annum, the strike price to \$13.00, the current stock price to \$10.00, and the volatility to 20% per annum, the Black-Scholes option price for a European call option is found to be \$0.20, the same option price generated in the Monte Carlo simulation. As expected, both option pricing techniques yield the same answer, because Equation (3) is not mathematically different from discounting (under risk neutral pricing) the average of the simulated data for the option payoffs.

Exhibit 6 compares the two techniques to demonstrate where they are the same and where they take different paths to the same answer.

Monte Carlo simulation lets us visualize what is occurring in the Black-Scholes option pricing equation without knowing the stochastic calculus necessary to generate the result. Although the two results appear dissimilar because of the use of transformations within the Black-Scholes framework, Equation (3) is really the continuous-time equivalent of a Monte Carlo simulation. The series of transformations is necessary only to simplify the final answer, and should not be viewed as altering what is stated in Equation (3).

The stochastic calculus used to generate the Black-Scholes equation should not be made more complicated than necessary. Ultimately, the final result is the product of an algorithm with three partial derivatives (Ito's Lemma) and a particular transformation (making a normal distribution into a standard normal distribution) performed three times.

IV. Conclusion

We demonstrate the statistical nature of stochastic processes, risk neutral pricing, and Monte Carlo simulation to enable visualization of how an option is priced when the underlying stock price follows geometric Brownian motion. We introduce Ito's Lemma and a technique for transforming a normal distribution into a standard normal distribution in order to apply stochastic calculus to the same risk-neutralized GBM process to produce the Black-Scholes (1973) option pricing model.

Although both option pricing methods are equivalent, the literature does not usually help us connect the two techniques intuitively. We would suggest that seeing the linkage is essential for an understanding of option pricing in its more complex forms.

Because the transformations necessary for evaluating the integral lead to a formula that does not appear to be an integral equation, it is understandable why the connection between the two option pricing methods is lost when viewing only the final Black-Scholes solution. The main contribution of this paper is in the demonstration of the equivalence of the two techniques: more specifically, how Monte Carlo analysis aids in visualizing the initial integral equation of the Black-Scholes framework.■

[9]The accuracy of the Monte Carlo solution can be increased by updating S more frequently within each simulation trial.

Exhibit 6. A Comparison Between Black-Scholes Option Pricing and Monte Carlo Simulation Option Pricing

Black-Scholes Model	Monte Carlo Simulation
A GBM process for the future stock price is specified and risk-neutralized by introducing the risk-free rate into the mean process.	A GBM process for the future stock price is specified and risk-neutralized by introducing the risk-free rate into the mean process.
Transform the risk-neutral GBM stock price process into a return process that is ABM. Thus, S_T becomes $S_0\,e^R$ and all mathematical relationships are transformed into terms of R.	Generates random draws from the risk-neutral GBM stock price process over many trials, and updates the stock price frequently within each simulation trial.
The probability of a particular future price is represented by the probability of the return associated with the price.	The frequency of a particular future price within the trials is commensurate with the probability of the price within the specified GBM process.
The future option payoffs are calculated within the integral in terms of R.	From the simulated future prices, the future option payoffs are calculated.
By integrating the option payoffs over the entire distribution of returns and then discounting the result using the risk-free rate, the option price is computed.*	By taking the average of the option payoffs over all the simulated outcomes and discounting the result using the risk-free rate, the option price is computed.

*The integral can be simplified by adjusting the integral limits and by transformation to a standard normal distribution.

Appendix

Given Equation (4) in the text, the first step is to adjust the integral limits to correspond with the maximization function. To make this adjustment to the integral, the arguments within the maximization function need to be in terms of R. Thus, because the maximization requires: $S_0e^R \geq X$, solving in terms of R produces $R \geq ln(X / S_0)$. Correspondingly:

$$U = \infty$$

$$L = ln\left(X/S_0\right) \tag{A.1}$$

Further, when the option is valuable, its value is $(S_0 {*} e^R - X)$.

$$\int_L^U [e^{-R_F T}][S_0 e^R - X]\frac{1}{\sigma\sqrt{2\pi T}} exp\left(-\left[R-\left(R_F T-\frac{1}{2}\sigma^2 T\right)\right]^2 \Big/ 2\sigma^2 T\right) dR \tag{A.2}$$

$$U = \infty$$

$$L = ln\left(X/S_0\right)$$

Split the integral based on S_0e^R and X, because each component will combine with the normal probability distribution differently:

$$\int_{L_1}^{U_1} [e^{-R_F T}][S_0 e^R]\frac{1}{\sigma\sqrt{2\pi T}} exp\left(-\left[R-\left(R_F T-\frac{1}{2}\sigma^2 T\right)\right]^2 \Big/ 2\sigma^2 T\right) dR \tag{A3}$$

$$-\int_{L_2}^{U_2} [e^{-R_F T}][X]\frac{1}{\sigma\sqrt{2\pi T}} exp\left(-\left[R-\left(R_F T-\frac{1}{2}\sigma^2 T\right)\right]^2 \Big/ 2\sigma^2 T\right) dR \tag{A.4}$$

$$U_1 = U_2 = \infty$$

$$L_1 = L_2 = ln\left(X/S_0\right)$$

Focusing on the integral in Equation (A.3), recall that dR is distributed $N([R_F - (½)\sigma^2]T,\ \sigma^2T)$; recall also that dt has been set to the option maturity T. Define Q as $(R - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5}$. Solve for R in terms of Q: $R = Q\sigma(T)^{0.5} + [R_F - (½)\sigma^2]T$ and calculate $dR/dQ = \sigma(T)^{0.5}$ to solve for dR in terms of dQ: $dR = \sigma(T)^{0.5}dQ$. Next, reset the upper and lower limits in terms of Q. For the upper limit, $R = \infty = Q\sigma(T)^{0.5} + [R_F - (½)\sigma^2]T$. Solving for Q in terms of the upper limit, ∞, sets the upper limit to $Q = (\infty - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5} = \infty$. For the lower limit, $R = ln(S_0/X) = Q\sigma(T)^{0.5} + [R_F - (½)\sigma^2]T$. Solving for Q in terms of the lower limit, $ln(S_0/X)$, sets the lower limit to $Q = (ln(S_0/X) - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5}$.

Making the substitutions for R and dR in terms of Q and dQ and resetting the limits makes the integral in Equation (A.3):

$$S_0[e^{-R_F T}]\int_{L_1}^{U_1} exp\left(Q\sigma\sqrt{T} + \left[R_F - \left(1/2\right)\sigma^2\right]T\right)\frac{1}{\sqrt{2\pi}} exp\left(-Q^2/2\right) dQ \tag{A5}$$

$$U_1 = \infty$$

$$L_1 = \frac{ln\left(X/S_0\right) - \left[R_F - \left(1/2\right)\sigma^2\right]T}{\sigma\sqrt{T}}$$

The integral is simplified as terms with the risk-free rate in the exponent cancel and all other exponential terms are collected:

$$S_0 \int_{L_1}^{U_1} \frac{1}{\sqrt{2\pi}} exp\left(-\left[Q^2 - 2Q\sigma\sqrt{T} + \sigma^2 T\right]\Big/2\right) dQ$$
$$= S_0 \int_{L_1}^{U_1} \frac{1}{\sqrt{2\pi}} exp\left(-\left[Q - \sigma\sqrt{T}\right]^2\Big/2\right) dQ \quad (A.6)$$
$$U_1 = \infty$$
$$L_1 = \frac{ln\left(X/S_0\right) - \left[R_f - \left(\tfrac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}}$$

Because Q is not quite a standard normal distribution, define M as $(Q - \sigma T^{0.5})$ and redefine the integral again: $Q = M + \sigma T^{0.5}$, making $dM/dQ = 1$ or $dM = dQ$. The upper limit, $Q = \infty = M + \sigma T^{0.5}$, in terms of M, $M = \infty - \sigma T^{0.5} = \infty$, remains unchanged. The lower limit, $Q = (ln(S_0/X) - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5} = M + \sigma T^{0.5}$, in terms of M, $M = (ln(S_0/X) - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5} - \sigma T^{0.5} = (ln(S_0/X) - [R_F + (½)\sigma^2]T) \div \sigma(T)^{0.5}$, adjusts slightly. Again, reset the integral by substituting for Q and dQ and adjusting the limits.

$$S_0 \int_{L_1}^{U_1} \frac{1}{\sqrt{2\pi}} \exp\left(-M^2\big/2\right) dM \quad (A.7)$$
$$U_1 = \infty$$
$$L_1 = \frac{ln\left(X/S_0\right) - \left[R_f + \left(\tfrac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}}$$

Finally, M is distributed $N(0, 1)$ which is symmetric. Consequently, taking an integral from x to ∞ is the same as taking the integral from $-\infty$ to $-x$. Again, because M is "standard normal", tables exist for the (cumulative) standard normal distribution that evaluate the integral from $-\infty$ up to a specified value. To take advantage of this convenience, invert the integral and define the term d_1, which is the negative of the lower limit in Equation (A.7):

$$S_0 \int_{L_1}^{U_1} \frac{1}{\sqrt{2\pi}} \exp\left(-M^2\big/2\right) dM \quad (A.8)$$
$$U_1 = \frac{ln\left(S_0/X\right) + \left[R_f + \left(\tfrac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}} = d_1$$
$$L_1 = -\infty$$

In Black-Scholes terminology, the integral becomes: $S_0 N(d_1)$, where $N(*)$ represents the standard normal cumulative distribution function.

The second integral Equation (A.4) is much easier to evaluate. Take the present value of the strike price outside the integral and again define Q as $(R - [R_F - (½)\sigma^2]T) \div \sigma(T)^{0.5}$. Solve for R in terms of Q: $R = Q\sigma(T)^{0.5} + [R_F - (½)\sigma^2]T$ and calculate $dR/dQ = \sigma(T)^{0.5}$ to solve for dR in terms of dQ: $dR = \sigma(T)^{0.5} dQ$. Adjust the limits of the integral equation in the same manner as before and make the appropriate substitutions for R and dR in terms of Q and dQ.

$$-Xe^{-R_f T} \int_{L_2}^{U_2} \frac{1}{\sqrt{2\pi}} \exp\left(-Q^2\big/2\right) dQ \quad (A.9)$$
$$U_2 = \infty$$
$$L_2 = \frac{ln\left(X/S_0\right) - \left[R_f - \left(\tfrac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}}$$

Because Q is standard normal, a second transformation is not necessary. Invert the integral and define the term d_2 as:

$$-Xe^{-R_f T} \int_{L_2}^{U_2} \frac{1}{\sqrt{2\pi}} \exp\left(-Q^2\big/2\right) dQ \quad (A.10)$$
$$U_2 = \frac{ln\left(S_0/X\right) + \left[R_f - \left(\tfrac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}} = d_2$$
$$L_2 = -\infty$$

In Black-Scholes terminology, the integral becomes: $-X\ exp(-R_F T)\ N(d_2)$. The complete Black-Scholes equation for a call option becomes: $S_0 N(d_1) - X\ exp(-R_F T)\ N(d_2)$.[10]

[10]For more discussion on this particular derivation of the Black-Scholes model, see Neftci (2000), Chapter 15.

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